

# THE 2-ADIC VALUATION OF A SEQUENCE ARISING FROM A RATIONAL INTEGRAL

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**ABSTRACT.** We analyze properties of the 2-adic valuations of an integer sequence that originates from an explicit evaluation of a quartic integral. We also give a combinatorial interpretation of the valuations of this sequence. Connections with the orbits arising from the Collatz problem are discussed.

## 1. INTRODUCTION

The sequence

$$(1.1) \quad A_{l,m} = \frac{l! m!}{2^{m-l}} \sum_{k=l}^m 2^k \binom{2m-2k}{m-k} \binom{m+k}{m} \binom{k}{l}$$

for  $m \in \mathbb{N}$  and  $0 \leq l \leq m$  appears in the evaluation of the definite integral

$$(1.2) \quad N_{0,4}(a; m) = \int_0^\infty \frac{dx}{(x^4 + 4ax^2 + 1)^{m+1}}.$$

Explicitly,

$$(1.3) \quad N_{0,4}(a; m) = \frac{\pi}{\sqrt{2} m! (4(2a+1))^{m+1/2}} \sum_{l=0}^m A_{l,m} \frac{a^l}{l!}.$$

The evaluation of  $A_{l,m}$  using (1.1) is efficient if  $l$  is close to  $m$ . For instance,

$$(1.4) \quad A_{m,m} = 2^m (2m)! \text{ and } A_{m-1,m} = 2^{m-1} (2m-1)! (2m+1).$$

In [1] it is shown that  $A_{l,m}$  is always an integer. An efficient method for the evaluation of these sequences when  $l$  is small is presented there. For example,

$$(1.5) \quad A_{0,m} = \prod_{k=1}^m (4k-1) \text{ and } A_{1,m} = (2m+1) \prod_{k=1}^m (4k-1) - \prod_{k=1}^m (4k+1).$$

The results described in this paper started as empirical observations on the behavior of  $\nu_2(A_{l,m})$ , the 2-adic valuation of  $A_{l,m}$ . Recall that  $\nu_2(x)$  is the highest power of 2 that divides  $x$ .

The 2-adic valuation of  $A_{0,m}$  follows directly from (1.5). Clearly  $A_{0,m}$  is odd, so  $\nu_2(A_{0,m}) = 0$ . The 2-adic valuation of  $A_{1,m}$  is given by

$$(1.6) \quad \nu_2(A_{1,m}) = \nu_2(m(m+1)) + 1.$$

This is the main result of [1].

The first goal of this paper is to present the following generalization of (1.6).

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*Date:* February 1, 2008.

*1991 Mathematics Subject Classification.* Primary 11B50, Secondary 05A15.

*Key words and phrases.* valuations, compositions, generating functions.

**Theorem 1.1.** *The 2-adic valuation of  $A_{l,m}$  satisfies*

$$(1.7) \quad \nu_2(A_{l,m}) = \nu_2((m+1-l)_{2l}) + l,$$

where  $(a)_k = a(a+1)\cdots(a+k-1)$  is the Pochhammer symbol.

As a consequence of this theorem we prove some interesting combinatorial properties of the sequence  $A_{l,m}$ . Henceforth, we assume that the index  $l \in \mathbb{N}$  is fixed and  $m \geq l$ .

Figure 1 shows the graph of  $\nu_2(A_{60,m})$  for  $60 \leq m \leq 450$ . The horizontal axis is the translate  $m' = m - 59$ , so the indexing starts at 1.

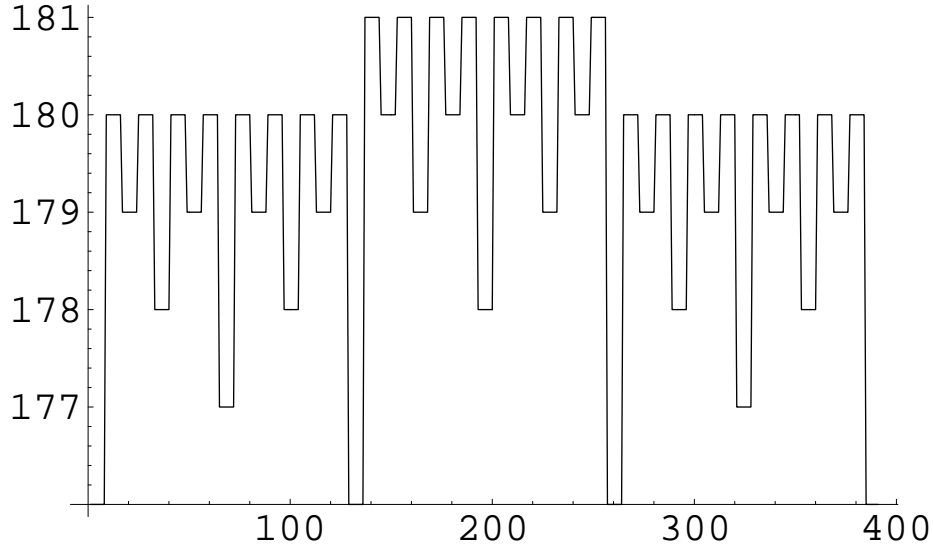


FIGURE 1. The 2-adic valuation of  $A_{60,m}$  for  $1 \leq m' \leq 400$

The figure suggests that the values of  $\{\nu_2(A_{60,m}) : m \geq 60\}$  have a *block structure* meaning that they are composed of consecutive blocks, all of the same length. Indeed, this sequence begins with

$$\{176, 176, 176, 176, 176, 176, 176, 176, 180, 180, 180, 180, 180, 180, 180, 180, 179, 179, 179, 179, 179, 179, 179, 179, 180, 180, 180, 180, 180, 180, 180, 180, \dots\},$$

which is formed by blocks of length 8.

This motivates the next definition.

**Definition 1.2.** Let  $s \in \mathbb{N}$ ,  $s \geq 2$ . We say that a sequence  $\{a_j : j \in \mathbb{N}\}$  is *simple of length  $s$*  ( or *s-simple*) if  $s$  is the largest integer such that, for each  $t \in \{0, 1, 2, \dots\}$ , we have

$$(1.8) \quad a_{st+1} = a_{st+2} = \dots = a_{s(t+1)}.$$

The sequence  $\{a_j : j \in \mathbb{N}\}$  is said to have a *block structure* if it is *s-simple* for some  $s \geq 2$ .

Using theorem 1.1 we evaluate

$$(1.9) \quad \nu_2(A_{l,m+1}) - \nu_2(A_{l,m}) = \nu_2(m+l+1) - \nu_2(m-l+1).$$

We then use this fact to establish in Theorem 3.5 that the sequence of integers  $\{\nu_2(A_{l,m}) : m \geq l\}$  is  $2^{1+\nu_2(l)}$ -simple.

The combinatorial properties of the sequence  $\nu_2(A_{l,m})$  are described as an algorithm:

**The maps  $F$  and  $T$ .** Consider the operators defined on sequences by:

$$(1.10) \quad F(\{a_1, a_2, a_3, \dots\}) := \{a_1, a_1, a_2, a_3, \dots\},$$

and

$$(1.11) \quad T(\{a_1, a_2, a_3, \dots\}) := \{a_1, a_3, a_5, a_7, \dots\}.$$

Now introduce the sequence  $c$  as

$$(1.12) \quad c := \{\nu_2(m) : m \geq 1\} = \{0, 1, 0, 2, 0, 1, 0, 3, 0, \dots\}.$$

**The algorithm:**

- 1) Start with the sequence  $X(l) := \{\nu_2(A_l(l+m-1)) : m \geq 1\}$ .
- 2) Find  $n \in \mathbb{N}$  so that the sequence  $X(l)$  is  $2^n$ -simple. Define  $Y(l) := T^n(X(l))$ . At the initial stage, Theorem 3.5 ensures that  $n = 1 + \nu_2(l)$ .
- 3) Introduce the shift  $Z(l) := Y(l) - c$ .
- 4) Define  $W(l) := F(Z(l))$ .

If  $W$  is a constant sequence, then STOP; otherwise go to step 2) with  $W$  instead of  $X$ . Define  $X_k(l)$  as the new sequence at the end of the  $(k-1)$ th cycle of this process, with  $X_1(l) = X(l)$ .

The next theorem justifies that the steps described above make sense and comprise an algorithm. In Section 5, we prove a stronger result (as Theorem 5.3) which states that the algorithm finishes in a finite number of steps and that  $W(l)$  is essentially  $X(j)$ , for some  $j > l$ . This will readily imply Theorem 1.3 as a direct consequence.

**Theorem 1.3.** *For general  $k \in \mathbb{N}$ , the sequence  $X_k(l)$  is  $2^{n_k}$ -simple for some  $n_k \in \mathbb{N}$ .*

**Note.** The operators  $F$  and  $T$ , defined in (1.10) and (1.11) respectively, play an important role in the proof of this conjecture.

**Definition 1.4.** Let  $\omega(l)$  be the number of steps required for the algorithm to yield a constant sequence. The sequence of integers

$$(1.13) \quad \Omega(l) := \{n_1, n_2, n_3, \dots, n_{\omega(l)}\}$$

is called the *reduction sequence* of  $l$ . The number  $\omega(l)$  will be called the *reduction length* of  $l$ . The constant sequence obtained after  $\omega(l)$  steps is called the *reduced constant*.

TABLE 1. Reduction sequence for  $1 \leq l \leq 15$ .

$l$	binary form	$\Omega(l)$
4	100	3
5	101	1, 2
6	110	2, 1
7	111	1, 1, 1
8	1000	4
9	1001	1, 3
10	1010	2, 2
11	1011	1, 1, 2
12	1100	3, 1
13	1101	1, 2, 1
14	1110	2, 1, 1
15	1111	1, 1, 1, 1

In Corollary 5.6 we enumerate  $\omega(l)$  as the number of ones in the binary expansion of  $l$ . Therefore the algorithm yields a constant sequence in a finite number of steps. In fact, the algorithm terminates in  $O(\log_2(l))$  steps.

Table 1 shows the results of the algorithm for  $4 \leq l \leq 15$ .

We also provide a combinatorial interpretation of  $\Omega(l)$ . This requires the composition of the index  $l$ .

**Definition 1.5.** Let  $l \in \mathbb{N}$ . The *composition* of  $l$ , denoted by  $\Omega_1(l)$ , is defined as follows: write  $l$  in binary form. Read the sequence from right to left. The first part of  $\Omega_1(l)$  is the number of digits up to and including the first 1 read in the corresponding binary sequence; the second one is the number of additional digits up to and including the second 1 read, and so on.

For example,

$$(1.14) \quad \Omega_1(13) = \{1, 2, 1\} \text{ and } \Omega_1(14) = \{2, 1, 1\}.$$

Observing the values in Table 1,  $\Omega_1(13) = \Omega(13)$  and  $\Omega_1(14) = \Omega(14)$ . We claim that this is always true.

**Theorem 1.6.** *The reduction sequence  $\Omega(l)$  associated to an integer  $l$  is the sequence of compositions of  $l$ , that is,*

$$(1.15) \quad \Omega(l) = \Omega_1(l)$$

This assertion is slightly restated and proved in Section 4, as 5.4. See the Note following the latter theorem.

## 2. THE 2-ADIC VALUATIONS OF $A_{l,m}$

We now present two proofs of theorem 1.1.

*Proof. First proof.* Define the numbers

$$(2.1) \quad B_{l,m} := \frac{A_{l,m}}{2^l(m+1-l)_{2l}}.$$

We need to prove that  $B_{l,m}$  is odd. The WZ-method [6] shows that the numbers  $B_{l,m}$  satisfy the recurrence

$$B_{l-1,m} = (2m+1)B_{l,m} - (m-l)(m+l+1)B_{l+1,m}, \quad 1 \leq l \leq m-1.$$

The initial values  $B_{m,m} = 1$  and  $B_{m-1,m} = 2m+1$  show that  $B_{l,m}$  is an odd integer as required.

**Second proof.** We have

$$(2.2) \quad \nu_2(A_{l,m}) = l + \nu_2 \left( \sum_{k=l}^m T_{m,k} \frac{(m+k)!}{(m-k)!(k-l)!} \right),$$

where

$$(2.3) \quad T_{m,k} = \frac{(2m-2k)!}{2^{m-k}(m-k)!}.$$

The identity

$$(2.4) \quad T_{m,k} = \frac{(2(m-k))!}{2^{m-k}(m-k)!} = (2m-2k-1)(2m-2k-3) \cdots 3 \cdot 1$$

shows that  $T_{m,k}$  is an odd integer. Then (2.2) can be written as

$$\begin{aligned} \nu_2(A_{l,m}) &= l + \nu_2 \left( \sum_{k=0}^{m-l} T_{m,l+k} \frac{(m+k+l)!}{(m-k-l)!k!} \right) \\ &= l + \nu_2 \left( \sum_{k=0}^{m-l} T_{m,l+k} \frac{(m-k-l+1)_{2k+2l}}{k!} \right). \end{aligned}$$

The term corresponding to  $k=0$  is singled out as we write

$$\nu_2(A_{l,m}) = l + \nu_2 \left( T_{m,l}(m-l+1)_{2l} + \sum_{k=1}^{m-l} T_{m,l+k} \frac{(m-k-l+1)_{2k+2l}}{k!} \right).$$

The claim

$$(2.5) \quad \nu_2 \left( \frac{(m-k-l+1)_{2k+2l}}{k!} \right) > \nu_2((m-l+1)_{2l})$$

for any  $k$ ,  $1 \leq k \leq m-l$ , will complete the proof.

To prove (2.5) we use the identity

$$\frac{(m-k-l+1)_{2k+2l}}{k!} = (m-l+1)_{2l} \cdot \frac{(m-l-k+1)_k (m+l+1)_k}{k!}$$

and the fact that the product of  $k$  consecutive numbers is always divisible by  $k!$ .

This follows from the identity

$$(2.6) \quad \frac{(a)_k}{k!} = \binom{a+k-1}{k}.$$

Now if  $m+l$  is odd,

$$(2.7) \quad \nu_2 \left( \frac{(m-l-k+1)_k}{k!} \right) \geq 0 \text{ and } \nu_2((m+l+1)_k) > 0,$$

and if  $m + l$  is even

$$(2.8) \quad \nu_2\left(\frac{(m+l+1)_k}{k!}\right) \geq 0 \text{ and } \nu_2((m-l-k+1)_k) > 0.$$

This proves (2.5) and establishes the theorem.  $\square$

### 3. PROPERTIES OF THE FUNCTION $\nu_2(A_{l,m})$

In this section, we describe properties of the function  $\nu_2(A_{l,m})$  for  $l$  fixed and  $m \geq l$ . In particular, we show that each of these sequences has a block structure.

**Theorem 3.1.** *Let  $l \in \mathbb{N}$  be fixed. Then for  $m \geq l$ , we have*

$$(3.1) \quad \nu_2(A_{l,m+1}) - \nu_2(A_{l,m}) = \nu_2(m+l+1) - \nu_2(m-l+1).$$

*Proof.* From (1.7) and  $(a)_k = (a+k-1)!/(a-1)!$ , we have

$$(3.2) \quad \nu_2(A_{l,m}) = \nu_2\left(\frac{(m+l)!}{(m-l)!}\right) + l.$$

This implies

$$\begin{aligned} \nu_2(A_{l,m+1}) - \nu_2(A_{l,m}) &= \nu_2\left(\frac{(m+l+1)!}{(m-l+1)!}\right) - \nu_2\left(\frac{(m+l)!}{(m-l)!}\right) \\ &= \nu_2\left(\frac{(m+l+1)!}{(m-l+1)!} \cdot \frac{(m-l)!}{(m+l)!}\right) \\ &= \nu_2\left(\frac{m+l+1}{m-l+1}\right). \end{aligned}$$

The result follows from here.  $\square$

The next corollary is a special case of Theorem 3.1.

**Corollary 3.2.** *The sequence  $\nu_2(A_{l,m})$  satisfies*

1)  $\nu_2(A_{l,l+1}) = \nu_2(A_{l,l})$ .

2) For  $l$  even,

$$\nu_2(A_{l,l+3}) = \nu_2(A_{l,l+2}) = \nu_2(A_{l,l+1}) = \nu_2(A_{l,l}).$$

3) The sequence  $\nu_2(A_{1,m})$  is 2-simple, i.e.,  $\nu_2(A_{1,m+1}) = \nu_2(A_{1,m})$ . In fact,

$$A_{1,m} = \{2, 2, 3, 3, 2, 2, 4, 4, 2, 2, \dots\}.$$

Fix  $k, l \in \mathbb{N}$  and let  $\mu := 1 + \nu_2(l)$ . Define the sets

$$(3.3) \quad C_{k,l} := \{l + k \cdot 2^\mu + j : 0 \leq j \leq 2^\mu - 1\}.$$

Clearly the cardinality of  $C_{k,l}$  is  $2^\mu$ . For example, if  $l \in \mathbb{N}$  is odd, then  $\mu = 1$  and

$$(3.4) \quad C_{k,l} = \{l + 2k, l + 2k + 1\}.$$

The next result is immediate.

**Lemma 3.3.** *The sets  $\{C_{k,l} : k \geq 0\}$  form a disjoint partition; namely,*

$$(3.5) \quad \{m \in \mathbb{N} : m \geq l\} = \bigcup_{k \geq 0} C_{k,l},$$

and  $C_{r,l} \cap C_{t,l} = \emptyset$ , whenever  $r \neq t$ .

**Lemma 3.4.** *Fix  $l \in \mathbb{N}$ .*

1) *The sequence  $\{\nu_2(A_{l,m}) : m \in C_{k,l}\}$  is constant. We denote this value by  $\nu_2(C_{k,l})$ .*

2) *For  $k \geq 0$ ,  $\nu_2(C_{k+1,l}) \neq \nu_2(C_{k,l})$ .*

*Proof.* Suppose  $0 \leq j \leq 2^\mu - 2$ . Then

$$(3.6) \quad \nu_2(2l + k \cdot 2^\mu) \geq \nu_2(k \cdot 2^\mu) \geq \mu > \nu_2(j + 1),$$

and hence

$$(3.7) \quad \nu_2(2l + k \cdot 2^\mu + j + 1) = \nu_2(j + 1) = \nu_2(k \cdot 2^\mu + j + 1).$$

Using these facts and (3.1), we obtain

$$\begin{aligned} \nu_2(A_{l,l+k \cdot 2^\mu + j + 1}) - \nu_2(A_{l,l+k \cdot 2^\mu + j}) &= \nu_2(2l + k \cdot 2^\mu + j + 1) - \nu_2(k \cdot 2^\mu + j + 1) \\ &= \nu_2(j + 1) - \nu_2(j + 1) = 0 \end{aligned}$$

for consecutive values in  $C_{k,l}$ . This proves part 1). To prove part 2), it suffices to take elements  $l + k \cdot 2^\mu + 2^\mu - 1 \in C_{k,l}$  and  $l + (k + 1) \cdot 2^\mu \in C_{k+1,l}$  and compare their 2-adic values. Again by (3.1), we have

$$\begin{aligned} \nu_2(A_{l,l+(k+1) \cdot 2^\mu}) - \nu_2(A_{l,l+(k+1) \cdot 2^\mu - 1}) &= \nu_2(2l + (k + 1) \cdot 2^\mu) - \nu_2((k + 1) \cdot 2^\mu) \\ &= \mu + \nu_2(2l \cdot 2^{-\mu} + k + 1) - \mu - \nu_2(k + 1) \\ &= \nu_2(2l \cdot 2^{-\mu} + k + 1) - \nu_2(k + 1) \neq 0. \end{aligned}$$

The last step follows from  $2l \cdot 2^{-\mu}$  being odd and thus  $2l \cdot 2^{-\mu} + k + 1$  and  $k + 1$  having opposite parities. This completes the proof.  $\square$

**Theorem 3.5.** *For each  $l \geq 1$ , the set  $\{\nu_2(A_{l,m}) : m \geq l\}$  is an  $s$ -simple sequence, with  $s = 2^{1+\nu_2(l)}$ .*

*Proof.* From Lemma 3.3 and Lemma 3.4, we know that  $\nu_2(\cdot)$  maintains a constant value on each of the disjoint sets  $C_{k,l}$ . The length of each of these blocks is  $2^{1+\nu_2(l)}$ .  $\square$

#### 4. THE ALGORITHM AND ITS COMBINATORIAL INTERPRETATION

The proof of the Theorem 1.6 requires some preliminaries.

A) Given the values of  $\Omega_1(l)$  for  $2^j \leq l \leq 2^{j+1} - 1$ , the list for  $2^{j+1} \leq l \leq 2^{j+2} - 1$  is formed according to the following rule:

$l$  is even: add 1 to the first part of  $\Omega_1(l/2)$  to obtain  $\Omega_1(l)$ ;

$l$  is odd: prepend a 1 to  $\Omega_1(\frac{l-1}{2})$  to obtain  $\Omega_1(l)$ .

This is clear: if  $x_1 x_2 \cdots x_t$  is the binary representation of  $l$ , then  $x_1 x_2 \cdots x_t 0$  is the one for  $2l$ . Thus, the first part of  $\Omega_1(2l)$  is increased by 1, due to the extra 0 on the right. The relative position of the remaining 1s stays the same. A similar argument takes care of  $\Omega_1(2l + 1)$ . The extra 1 that is placed at the end of the binary representation gives the first 1 in  $\Omega_1(2l + 1)$ .

B) We now relate the 2-adic valuation of  $A_{l,m}$  to that of  $A_{\lfloor l/2 \rfloor, m}$ .

**Proposition 4.1.** *Let*

$$(4.1) \quad \lambda_l := \frac{1 - (-1)^l}{2}, \quad M_0 := \lfloor \frac{m + \lambda_l}{2} \rfloor.$$

*Then*

$$(4.2) \quad \nu_2(A_{l,m}) = 2l - \lfloor l/2 \rfloor + \lambda_l \nu_2(M_0 - \lfloor l/2 \rfloor) + \nu_2(A_{\lfloor l/2 \rfloor, M_0}).$$

*Proof.* We present the details for  $\nu_2(A_{2l, 2m})$ . Theorem 1.1 gives

$$\begin{aligned} \nu_2(A_{2l, 2m}) &= \nu_2((2m - 2l + 1)_{4l}) + 2l \\ &= \nu_2((2m - 2l + 1)(2m - 2l + 2) \cdots (2m + 2l - 1)(2m + 2l)) + 2l \\ &= \nu_2(2^{2l}(m - l + 1)(m - l + 2) \cdots (m + l)) + 2l \\ &= 4l + \nu_2((m - l + 1)_{2l}) \\ &= 3l + \nu_2(A_{l,m}). \end{aligned}$$

A repeated application deals with the general case.  $\square$

**Corollary 4.2.** *The 2-adic valuation of  $A_{l,m}$  satisfies*

$$(4.3) \quad \nu_2(A_{l,m}) = 2l + \nu_2(l!) + \sum_{k \geq 0} \lambda_{\lfloor l/2^k \rfloor} \nu_2(M_k - \lfloor l/2^{k+1} \rfloor)$$

*where*

$$(4.4) \quad M_k = \lfloor \frac{m + \lambda_l + 2\lambda_{\lfloor l/2 \rfloor} + \cdots + 2^k \lambda_{\lfloor l/2^k \rfloor}}{2^{1+k}} \rfloor = \lfloor \frac{m + \sum_{n=0}^k 2^n \lambda_{\lfloor l/2^n \rfloor}}{2^{1+k}} \rfloor.$$

*Proof.* This is a repeated application of Proposition 4.1. The first term results from

$$\begin{aligned} \sum_{k \geq 0} \left( 2 \lfloor \frac{l}{2^k} \rfloor - \lfloor \frac{l}{2^{k+1}} \rfloor \right) &= 2l + \sum_{k \geq 1} \lfloor \frac{l}{2^k} \rfloor \\ &= 2l + \nu_2(l!). \end{aligned}$$

$\square$

## 5. VERIFICATION OF THE ALGORITHM AND THE REDUCTION SEQUENCE

In this section we establish Theorems 1.3 and 1.6, strengthening as Theorem 5.3 and restated as Theorem 5.4, respectively. First we prove that the reduction process alluded to in the Introduction is in fact an algorithm. This will be followed by a proof that the reduction sequence that comes from completing the algorithm on  $X(l)$  is identical to the composition sequence of the integer  $l$ .

We now remind the reader of some definitions and nomenclature:  $\Omega(l)$  is the reduction sequence of  $X(l)$ , and  $\Omega_1(l)$  is the composition of the integer  $l$ . Also,  $X_k(l)$  is the new sequence outputted at the end of the  $(k - 1)$ th cycle of the algorithm, and we also use the previously defined operators

$$F(\{a_1, a_2, a_3, \dots\}) = \{a_1, a_1, a_2, a_3, \dots\}$$

and

$$T(\{a_1, a_2, a_3, \dots\}) = \{a_1, a_3, a_5, a_7, \dots\}.$$



Observe that  $T(\{a_m : m \geq 1\}) = \{a_{2m-1} : m \geq 1\}$ . Recall the constant sequence

$$c := \{\nu_2(m) : m \geq 1\} = \{0, 1, 0, 2, 0, 1, 0, 3, 0, \dots\}.$$

**Convention:** We write  $A_{l,m}$  and  $A_l(m)$  interchangeably.

**Notations:** Bold-face letters will denote constant sequences, as in,  $\mathbf{t} = \{t, t, t, \dots\}$ . The initial sequence is  $X(l) = \{\nu_2(A_l(m-1+l)) : m \geq 1\}$ . Note from Theorem 1.1 that

$$X(l) = \{\nu_2\left(\frac{(m-1+2l)!}{(m-1)!}\right) + l : m \geq 1\}.$$

**Definition 5.1.** A sequence  $\mathbf{a} = \{a_1, a_2, a_3, \dots\}$  is a *translate* of  $\mathbf{b} = \{b_1, b_2, b_3, \dots\}$  if  $\mathbf{a} = \mathbf{b} + \mathbf{t}$ , for some constant sequence  $\mathbf{t}$ .

Now, before proving the next main result, we consider the base case  $l = 1$ .

**Lemma 5.2.** *The initial case  $l = 1$  satisfies*

$$(5.1) \quad W(1) = F(T(X(1)) - c) = \mathbf{2}.$$

*Proof.* Since  $\nu_2(A_1(m)) = \nu_2(m(m+1)) + 1$  and  $\nu_2(2m-1) = 0$ , we have

$$T(X(1)) = \{\nu_2((2m-1)(2m)) + 1 : m \geq 1\} = \{\nu_2(m) + 2 : m \geq 1\} = c + \mathbf{2}.$$

Then the assertion follows from  $F(\mathbf{t}) = \mathbf{t}$  for a constant  $\mathbf{t}$ .  $\square$

Remember now that  $X(l)$  is  $2^n$ -simple, hence so are its' translates. Thus, the next result will suffice to prove Theorem 1.3.

**Theorem 5.3.** *The algorithm terminates after finite iterations. Further, in each cycle,  $W(l)$  is a translate of  $X(j)$ , for some  $j < l$ .*

*Proof.* Start by rewriting the terms in  $X(l)$  as

$$\nu_2\left(\frac{(m-1+2l)!}{(m-1)!}\right) + l = \nu_2((m-1+2l)(m-2+2l) \cdots (m+1)m), \quad m \geq 1.$$

Then, the operator  $T$  acts on these to yield (for  $m \geq 1$ )

$$\begin{aligned} & \nu_2((2m-2+2l)(2m-3+2l) \cdots (2m)(2m-1)) + l \\ &= \nu_2((m-1+l) \cdots (m)) + 2l \\ (5.2) \quad &= \nu_2\left(\frac{(m-1+l)!}{(m-1)!}\right) + 2l. \end{aligned}$$

**Case I:**  $l$  is even. From (5.2), we can easily obtain the relation (with  $l_2 = l/2$ )

$$T(X(l)) = \{\nu_2\left(\frac{(m-1+2l_2)!}{(m-1)!}\right) + l_2 + t : m \geq 1\} = X(l_2) + \mathbf{t}, \quad t = 3l_2.$$

**Case II:**  $l$  is odd. Upon subtracting the sequence  $c = \{\nu_2(m) : m \geq 1\}$  from (5.2) and letting  $l_1 = (l-1)/2$ , we get that

$$\nu_2\left(\frac{(m+l-1)!}{m!}\right) + 2l = \nu_2\left(\frac{(m+2l_1)!}{m}\right) + 2l = \nu_2\left(\frac{(m+2l_1)!}{m!}\right) + l_1 + (3l_1+2),$$

for  $m \geq 1$ . Finally, apply the operator  $F$  to the last sequence and find

$$W(l) = \left\{ \nu_2 \left( \frac{(m-1+2l_1)!}{(m-1)!} \right) + l_1 + t : m \geq 1 \right\} = X(l_1) + \mathbf{t}, \quad t = 3l_1 + 2.$$

Here, we have utilized the fact that  $\nu_2(2l + (l-1)!) = \nu_2(2l + l!) = 1$  which is valid for  $l$  odd. This justifies that the first term augmented in the sequence, as a result of the action of  $F$ , coincides with the next term (these are values at  $m = 1$  and  $m = 2$ , respectively, in  $X(l_1)$ ).

We can now conclude that in either of the two cases (or a combination thereof), the index  $l$  shrinks dyadically as  $l_1$  or  $l_2$ . Thus the reduction algorithm must end in a finite step into a translate of  $X(1)$ . Since Lemma 5.2 handles  $X(1)$ , the proof is completed.  $\square$

**Theorem 5.4.** *Let  $\{k_1, \dots, k_n : 0 \leq k_1 < k_2 < \dots < k_n\}$ , be the unique collection of distinct positive integers such that*

$$(5.3) \quad l = \sum_{i=1}^n 2^{k_i}.$$

*Then the reduction sequence of  $l$  is  $\{k_1 + 1, k_2 - k_1, \dots, k_n - k_{n-1}\}$ .*

**Note.** The argument of the proof is to check that the rules of formation for  $\Omega_1(l)$  also hold for the reduction sequence  $\Omega(l)$ . This will incidentally elaborates the connection with 1.6. The proof is divided according to the parity of  $l$ .

*Proof.* The case  $l$  odd starts with  $l = 1$ , where the block length is 2. From Theorem 1.1 we obtain a constant sequence after iterating the algorithm once. Thus the algorithm terminates and the reduction sequence for  $l = 1$  is  $\Omega(1) = \{1\}$ .

Now consider the general even case:  $X(2l)$ . Applying  $T$  to this sequence yields a translate of  $X(l)$  by Theorem 5.3; this does not affect the reduction sequence  $\Omega(l)$ , but the doubling of block length increases the first term of  $\Omega(l)$  by 1. Therefore

$$(5.4) \quad \Omega(2l) = \{k_1 + 2, k_2 - k_1, \dots, k_n - k_{n-1}\}.$$

This is precisely what happens to the binary digits of  $l$ : if

$$l = \sum_{i=1}^n 2^{k_i}, \text{ then } 2l = \sum_{i=1}^n 2^{k_i+1}.$$

This concludes the argument for even indices.

For the general odd case,  $X(2l+1)$ , we apply  $T$ , subtract  $c$  and then apply  $F$ . Again, by Theorem 5.3, this gives us a translate of  $X(l)$ . We conclude that, if the reduction sequence of  $l$  is

$$(5.5) \quad \{k_1 + 1, k_2 - k_1, \dots, k_n - k_{n-1}\},$$

then that of  $2l+1$  is

$$(5.6) \quad \{1, k_1 + 1, k_2 - k_1, \dots, k_n - k_{n-1}\}.$$

This is precisely the behavior of  $\Omega_1$ . The proof is complete.  $\square$

**Corollary 5.5.** *The reduced constant is  $2l + \nu_2(l!) = \nu_2(A_{l,l})$ .*

*Proof.* In Corollary 4.2, subtract the last term as per the reduction algorithm (or as implied by Theorem 5.4 or Theorem 1.6).  $\square$

**Corollary 5.6.** *The set  $\Omega(l)$  has cardinality*

$$(5.7) \quad s_2(l) = \text{the number of ones in the binary expansion of } l.$$

**Remarks:**

Write  $l$  in the binary form:  $l = \sum_{j=1}^n 2^{k_j}$  with  $0 \leq k_1 < \dots < k_n$ . Then, for the  $M_k$  defined in (4.4) can be rewritten as

$$M_{k_i} = \lfloor \frac{m + \sum_{j=1}^i 2^{k_j}}{2^{1+k_i}} \rfloor.$$

1) In light of this, Corollary 4.2 may be given in the form

$$(5.8) \quad \nu_2(A_{l,m}) = 2l + \nu_2(l!) + \sum_{i \geq 1} \nu_2(M_{k_i} - \lfloor l/2^{1+k_i} \rfloor).$$

2) Observe also that  $\nu_2(M_{k_i} - \lfloor l/2^{1+k_i} \rfloor)$  is a  $2^{1+k_i}$ -simple sequence, i.e. it has constant blocks of length  $2^{1+k_i}$ .

3) The sequence  $\nu_2(A_{l,m})$  inherits its  $2^{1+k_1}$ -simple structure from the term  $\nu_2(M_{k_1} - \lfloor l/2^{1+k_1} \rfloor)$ , which has the lowest period (or highest frequency) in the decomposition (5.8). Notice that this is consistent with Theorem 3.5, since  $k_1 = \nu_2(l)$ .

4) The sequence  $(\dots, \lambda_{\lfloor l/2 \rfloor}, \lambda_l)$  is the binary code for  $l$ , and  $(\dots, k_2 + 1, k_1 + 1)$  are the exponents of 2 in the binary format of  $2l$ .

5) For fixed  $l$ , we can construct the sequence  $\nu_2(A_{l,m})$  by reversing the algorithm. Write the binary code for  $2l = \sum_{j=1}^n 2^{1+k_j}$ , and then, starting with the  $\infty$ -simple (constant) sequence  $3l - s_2(l)$ , then add the  $2^{1+k_1}, 2^{1+k_2}, \dots, 2^{1+k_n}$ -simple sequences  $\nu_2(M_{k_i} - \lfloor l/2^{1+k_i} \rfloor)$ . Here, the successive differences  $(1+k_j) - (1+k_{j-1}) = k_j - k_{j-1}$ , for  $j = 1 = 1, \dots, n$ , encode the *period switching-gaps* (or *indices of sequence shifting* as compared to the preceding stages) on the one hand, and the *integer composition* of  $2l$  on the other. This shades more light into the bijective relationship between  $\Omega(l)$  and  $\Omega_1(l)$  that has been proven in Theorem 1.6.

**Note.** The function  $s_2(l)$  has recently appeared in a different divisibility problem. In these papers it is denoted by  $d(l)$ . Lengyel [5] conjectured, and De Wannemacker [7] proved, that the 2-adic valuation of the Stirling numbers of the second kind  $S(n, k)$  is given by

$$(5.9) \quad \nu_2(S(2^n, k)) = s_2(k) - 1.$$

The Stirling numbers are given by the identity

$$(5.10) \quad x^n = \sum_{k=0}^n S(n, k)x(x-1)(x-2)\cdots(x-k+1)$$

and they count the number of ways to partition a set with  $n$  elements into exactly  $k$  nonempty subsets. De Wannemacker [8] also established the inequality

$$(5.11) \quad \nu_2(S(n, k)) \geq s_2(k) - s_2(n), \quad 0 \leq k \leq n.$$

The study of the 2-adic valuation of Stirling numbers suggests that

$$(5.12) \quad \nu_2(S(2^n + 1, k + 1)) = s_2(k) - 1,$$

which is a companion of (5.9).

## 6. A CONNECTION WITH THE COLLATZ PROBLEM

The numbers

$$(6.1) \quad a_m := \nu_2(A_{1,m}) - 1 = \nu_2(m(m+1)),$$

given in (1.6), also appear in the well-known *Collatz* or  $3x + 1$  problem. Define a sequence by  $x_0(m) = m$  and let  $x_{k+1}(m) = T(x_k(m))$ , where

$$(6.2) \quad T(i) = \begin{cases} \frac{1}{2}i & \text{if } i \text{ is even,} \\ \frac{1}{2}(3i + 1) & \text{if } i \text{ is odd.} \end{cases}$$

The *orbit* of  $m \in \mathbb{N}$  is the set

$$(6.3) \quad \mathfrak{O}(m) := \{m, T(m), T^2(m), \dots\}.$$

The main conjecture for this problem is that *every orbit ends in the cycle*  $1 \rightarrow 2 \rightarrow 1$ . The reader will find in [3] an introduction to this problem and [2, 4] contain annotated bibliographies.

The connection with our work is given in the next theorem.

**Theorem 6.1.** *Let  $m \in \mathbb{N}$ . Then  $a_m := \nu_1(A_{1,m}) - 1 = \nu_2(m(m+1))$  is the first time at which the orbit  $\mathfrak{O}(m)$  changes parity. That is,*

$$(6.4) \quad m \equiv T(m) \equiv T^2(m) \equiv \dots \equiv T^{a_m-1}(m) \not\equiv T^{a_m}(m) \pmod{2}.$$

*Proof.* Suppose  $m$  is odd and write it as  $m = 2^j n - 1$ , with  $n$  odd. Then

$$(6.5) \quad j = \nu_2(m+1) \text{ and } n = \frac{m+1}{2^j}$$

are uniquely defined. Observe that

$$T(m) = T(2^j n - 1) = 3 \cdot 2^{j-1} n - 1$$

and for  $i < j$ ,

$$T^i(m) = T^i(2^j n - 1) = 3^i \cdot 2^{j-i} n - 1.$$

Finally,

$$T^j(m) = T^j(2^j n - 1) = 3^j n - 1.$$

To complete the proof, observe that

$$(6.6) \quad j = \nu_2(m+1) = \nu_2(m(m+1)) = N.$$

In the case  $m$  is even, write  $m = 2^t m_0$ , with  $m_0$  odd. Then

$$(6.7) \quad T^i(m) = 2^{t-i} m_0, \text{ for } 0 \leq i < t$$

and

$$(6.8) \quad T^t(m) = m_0.$$

The proof is completed by noticing that

$$(6.9) \quad t = \nu_2(m) = \nu_2(m(m+1)) = N.$$

□

For example take  $m = 63$ . Then  $x_1(63) = 95$ ,  $x_2(63) = 143$ ,  $x_3(63) = 215$ ,  $x_4(63) = 323$ ,  $x_5(63) = 485$ , and  $x_6(63) = 728$ . Thus,

$$(6.10) \quad \mathfrak{O}(63) = \{63, 95, 143, 215, 323, 485, \mathbf{728}, \dots\}.$$

It takes 6 iterations to produce an even entry. Observe that  $a_{63} = \nu_2((63)_2) = 6$ .

Similarly, we have

**Proposition 6.2.** *The first time the orbit of  $3^m - 1$  changes parity is after*

$$\nu_2(3^m(3^m - 1)) = \nu_2(3^m - 1) = \lambda_m + \nu_2(2m) = \nu_2(2^{1+\lambda_m}m)$$

*iterations.*

*Proof.* Use the binomial theorem for  $(2 + 1)^m - 1$ , while the generating function can be given by

$$(6.11) \quad \sum_{m \geq 1} \nu_2(3^m - 1)x^m = \frac{x^2}{1 - x^2} + \sum_{k \geq 0} \frac{x^{2^k}}{1 - x^{2^k}}.$$

□

## 7. A SYMMETRY CONJECTURE ON THE GRAPHS OF $\nu_2(A_{l,m})$

The graphs of the function  $\nu_2(A_{l,m})$ , where we take every other  $2^{1+\nu_2(l)}$ -element to reduce the repeating blocks to a single value, are shown in the next figures. We conjecture that these graphs have a symmetry property generated by what we call an *initial segment*: from which the rest is determined by adding a *central piece* followed by a *folding rule*. For example, in the case  $l = 1$ , the first few values of the reduced table are

$$\{2, 3, 2, 4, 2, 3, 2, 5, 2, 3, 2, \dots\}.$$

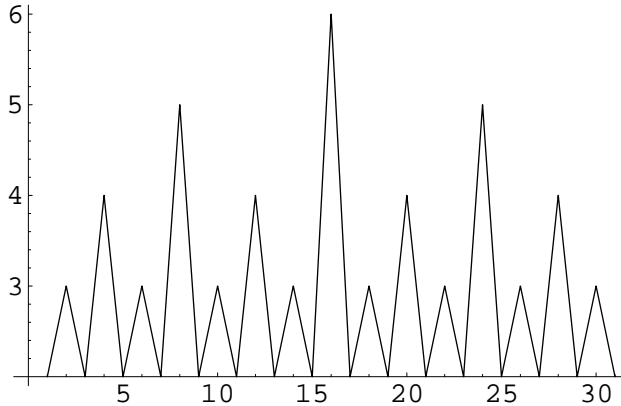


FIGURE 2. The 2-adic valuation of  $A_{1,m}$

The ingredients are:

*initial segment:*  $\{2, 3, 2\}$ ,

*central piece:* the value at the center of the initial segment, namely 3.

*rules of formation:* start with the initial segment and add 1 to the central piece and reflect.

This produces the sequence

$$\begin{aligned} \{2, 3, 2\} &\rightarrow \{2, 3, 2, 4\} \rightarrow \{2, 3, 2, 4, 2, 3, 2\} \rightarrow \{2, 3, 2, 4, 2, 3, 2, 5\} \rightarrow \\ &\rightarrow \{2, 3, 2, 4, 2, 3, 2, 5, 2, 3, 2, 4, 2, 3, 2\}. \end{aligned}$$

The details are shown in Figure 2.

We have found no way to predict the initial segment nor the central piece. Figure 3 shows the beginning of the case  $l = 9$ . From here one could be tempted to anticipate that this graph extends as in the case  $l = 1$ . This is not correct however, as can be seen in Figure 4. In fact, the initial segment is depicted in Figure 4 and its extension is shown in Figure 5.

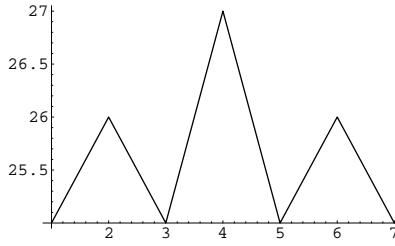


FIGURE 3. The beginning for  $l = 9$

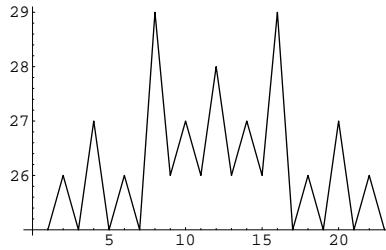
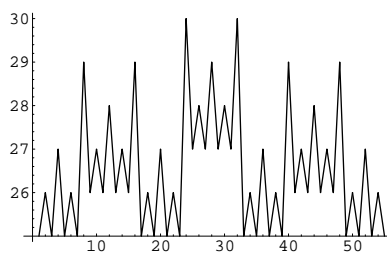
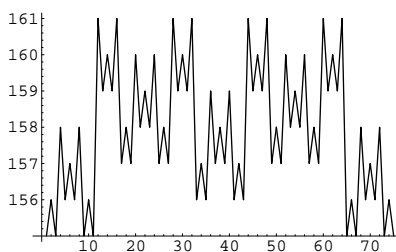


FIGURE 4. The continuation of  $l = 9$

The initial pattern can be quite elaborate. Figure 6 illustrates the case  $l = 53$ .

**Acknowledgements.** The last author acknowledges the partial support of nsf-dms 0409968. The second author was partially supported as a graduate student by the same grant.

FIGURE 5. The pattern for  $l = 9$  persistsFIGURE 6. The initial pattern for  $l = 53$ 

The work of the first author was done while visiting Tulane University in the Spring of 2006. The authors wish to thank Marc Chamberland for information on the  $3x+1$  problem and Aaron Jaggard for identifying our data with the composition sequence.

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